ON THE DIFFERENTIAL EQUATIONS OF THE EQUILIBRIUM OF AN INEXTENSIBLE STRING*

BY

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1. Introduction. In a recent communication on the problem of the rotating string or chain, I remarked the fact that of the great variety of problems on the equilibrium of strings there are few, if any, which treat cases other than those in which the field of force is rectilinear, namely parallel or central, and that consequently problems in which the string has a free end are too trivial, as far as the general question of equilibrium is concerned, to deserve mention.† The object of the present discussion is to enlarge upon that remark. It will appear in § 6 that there is a large class of cases other than rectilinear which may be explicitly integrated by quadratures.

Let it be assumed that the string lies in a plane field of force derivable from the potential V. The condition of equilibrium is then expressible by means of the calculus of variations as

$$\int \rho V ds \text{ a minimum}, \qquad \int ds \text{ a constant},$$

where ρ is the density.‡ The integral to render a minimum becomes

$$\int F ds = \int (\rho V + \lambda) ds \qquad (\lambda \text{ a parameter}). \tag{1}$$

This type of integral, where the integrand is the product of a function of position and the differential of arc, is of frequent occurrence in the applications of the calculus of variations. It includes the propagation of light in a medium, in which F is the index of refraction, and the brachistochrone problems, where F is the reciprocal of the velocity.

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† *The equilibrium of a heavy homogeneous chain in a uniformly rotating plane*, Annals of Mathematics, series 2, volume 9 (1908), pp. 99–115.

‡ The density may be considered as constant, although the case in which the density were a function of position (a condition rarely, if ever, realized in practice) is not excluded in the analysis; it would merely be necessary to replace F by ρF and note that the curves called equipotential in § 2 would be curves $\rho F =$ const.

The differential equation of the extremals in cartesian coördinates is

$$(2)\qquad \frac{d}{dx}\left(F\frac{y'}{\sqrt{1+y'^2}}\right)-\sqrt{1+y'^2}\,\frac{\partial F}{\partial y}=0$$

or

$$(3)\qquad Fy''=(F_y-F_xy')(1+y'^2).$$

This latter equation may be written in simpler form as

$$(4)\qquad y''+\phi_xy'^3-\phi_yy'^2+\phi_xy'-\phi_y=0,\qquad \phi=\log F.$$

These equations are of the second order and third degree. Such equations have the property that their degree as well as their order is unchanged by the general point transformation

$$(5)\qquad \bar{x}=X(x,y),\qquad \bar{y}=Y(x,y),$$

except that the resulting equation may be of lower degree owing to the vanishing of some of the coefficients in the transformed equation.

The question of the integrability of an equation of the second order may be viewed in several lights. From LIE's point of view, the question is as to the existence of 0, 1, 2, 3, or 8 independent infinitesimal transformations admitted by the equation.* In this direction LIE himself and his pupil TRESSE have carried on investigations in which the differential invariants figure prominently.† These methods are applicable to equations of higher degree than the third. On the other hand R. LIOUVILLE, without making use of LIE's methods but none the less by employing invariants of (5), has discussed this case of a cubic in y' in extended detail.‡ Leaving for the present the further consideration of these methods, I shall begin without introducing the theory of invariants.

If the equation $y''=\omega(x,y,y')$ is to admit the infinitesimal transformation

$$(6)\qquad Uf=\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}$$

the condition given by LIE §

$$(7)\qquad \begin{aligned}(\eta_y-2\xi_x-3\xi_yy')\omega-\xi_{yy}y'^3+(\eta_{yy}-2\xi_{xy})y'^2+(2\eta_{xy}-\xi_{xx})y'+\eta_{xx}\\ -\xi\omega_x-\eta\omega_y-[\eta_x+(\eta_y-\xi_x)y'-\xi_yy'^2]\,\omega_{y'}\equiv 0\end{aligned}$$

* See, for example, LIE–SCHEFFERS, *Differentialgleichungen*, especially Abteilung 5.

† TRESSE, *Sur les invariants différentielles des groupes continus de transformations*, Acta Mathematica, vol. 18 (1893), pp. 1–88, and *Détermination des invariants ponctuelles de l'equation ordinaire du second ordre* $y''=\omega(x,y,y')$, Jablonowski Preisschrift (1896).

‡ *Sur quelques équations différentielles non linéaires*, Journal de l'Ecole polytechnique, vol. 57 (1887), pp. 187–250, and *Mémoire sur les invariants de certaines équations différentielles et leurs applications*, ibid., vol. 59 (1889), pp. 7–76.

§ LIE–SCHEFFERS, *Differentialgleichungen*, Theorem 35, p. 363.

must be fulfilled identically in y'. This requires the four simultaneous linear partial differential equations

$$
(8)\qquad
\begin{aligned}
\eta_{xx} - 2\xi_x\phi_y + \eta_x\phi_x + \eta_y\phi_y - \xi\phi_{xy} - \eta\phi_{yy} &= 0\,,\\
\xi_{yy} - 2\eta_y\phi_x + \xi_y\phi_y + \xi_x\phi_x - \xi\phi_{xx} - \eta\phi_{xy} &= 0\,,\\
\eta_{yy} - 2\xi_{xy} + 2\xi_y\phi_x + 3\eta_x\phi_x - \eta_y\phi_y - \xi\phi_{xy} - \eta\phi_{yy} &= 0\,,\\
\xi_{xx} - 2\eta_{xy} + 2\eta_x\phi_y + 3\xi_y\phi_y - \xi_x\phi_x - \xi\phi_{xx} - \eta\phi_{xy} &= 0\,,
\end{aligned}
$$

of the second order in the two independent variables x, y and the two dependent variables ξ, η to have a common solution.

It is physically obvious and readily verified on equations (8) that, if F and consequently V and ϕ are functions of one variable alone, say x, there is a common solution $\xi = 0$, $\eta = 1$ which amounts to a translation along the equipotentials $V = \text{const.}$ In like manner if F is a function of $r = \sqrt{x^2 + y^2}$ alone there is the obvious common solution $\xi = -y$, $\eta = x$ which represents a rotation about the center of force at the origin. The complete integration of these two cases seems to have been familiar to JOHN BERNOULLI;* the results are

$$
(9)\qquad y = \int \frac{dx}{(c^2F^2 - 1)^{\frac{1}{2}}}, \qquad \theta = \int \frac{dr}{r(c^2F^2r^2 - 1)^{\frac{1}{2}}} \qquad (c \text{ a constant}),
$$

in terms of quadratures, from which it is easy to see cases that are and cases that are not such that the form of the curve can be expressed by trigonometric, hyperbolic, or elliptic functions.

2. Transformation to equipotential curves. If the reason for the integrability by quadratures in the foregoing cases be sought, it is seen to lie in the fact that the equation (2) in the first case, and the similar equation in polar coördinates in the second case, reduces to one term and is immediately integrable to

$$
(10)\qquad F\frac{y'}{\sqrt{1 + y'^2}} = \frac{1}{c}, \qquad F\frac{r^2\theta'}{\sqrt{1 + r^2\theta'^2}} = \frac{1}{c},
$$

wherein the variables are separable. Now, inasmuch as F is a function of V alone, a transformation to curvilinear coördinates u, v, in which one set of curves is the set of equipotentials and the other set is any independent family will take (1) into the form

$$
(11)\qquad \int F ds, \qquad ds = \sqrt{A du^2 + 2B du\, dv + C dv^2}
$$

where F is a function of V alone, and if the method of integration used in (10) and (9) is to be available it is necessary and sufficient that A, B, C be likewise functions of V only.

*See the numerous references in WALTON'S *Problems in Theoretical Mechanics*, third edition, Chapter 5.

Considerably to simplify the further calculation of these conditions and others that will arise it is desirable to transform from cartesian to minimal coördinates

$$(12)\quad m=x+iy,\quad n=x-iy;\qquad x=\frac{1}{2}(m+n),\quad y=\frac{1}{2i}(m-n);\qquad ds=\sqrt{dmdn}.$$

Then equations (2), (3), (4) become

$$(2')\qquad \frac{d}{dm}\left(F\frac{1}{2\sqrt{n'}}\right)-F_n\sqrt{n'}=0,$$

$$(3')\qquad Fn''=2F_m n'-2F_n n'^2,$$

$$(4')\qquad n''+\phi_n n'^2-\phi_m n'=0,\qquad \phi=\log F^2,$$

where it should be particularly noted that ϕ regarded as a function of m, n is the logarithm of the square of F regarded as a function of those variables instead of the logarithm of F. Incidentally the set of equation (8) becomes

$$(8')\qquad \begin{aligned} \eta_{mm}-\phi_m\eta_m&=0,\\ \xi_{nn}-\phi_n\xi_n&=0,\\ \eta_{nn}-2\xi_{mn}+\phi_n\eta_n-2\phi_m\xi_n+\phi_{mn}\xi+\phi_{nn}\eta&=0,\\ \xi_{mm}-2\eta_{mn}+\phi_m\xi_m-2\phi_n\eta_m+\phi_{mm}\xi+\phi_{mn}\eta&=0.\end{aligned}$$

On introducing the transformation to the equipotential curves, that is,

$$(13)\quad u=g(m,n),\quad v=h(m,n),\quad du=g_m dm+g_n dn,\quad dv=h_m dm+h_n dn,$$

where it is assumed that g is a function of $V(m,n)$, the integral to minimize is (11) where by virtue of

$$(14)\quad dm=\frac{h_n du-g_n dv}{\Delta},\quad dn=\frac{g_m dv-h_m du}{\Delta},\quad \Delta=\begin{vmatrix} g_m & g_n\\ h_m & h_n\end{vmatrix}\neq 0,\quad \int F\sqrt{dmdn},$$

the expressions for A, B, C are merely

$$(15)\qquad A=-\frac{h_m h_n}{\Delta^2},\qquad 2B=-\frac{g_m h_n+g_n h_m}{\Delta^2},\qquad C=-\frac{g_m g_n}{\Delta^2};$$

and these are, under the present hypotheses, to be functions of V and consequently of g. The conditions may be restated in better form.

It is evident that

$$\frac{2B}{A}=\frac{g_m h_n+g_n h_m}{h_m h_n}=\frac{g_m}{h_m}+\frac{g_n}{h_n}\qquad \text{is a function of } u=g(m,n),$$

$$\frac{2B}{C}=\frac{g_m h_n+g_n h_m}{g_m g_n}=\frac{h_n}{g_n}+\frac{h_m}{g_m}\qquad \text{is a function of } u=g(m,n).$$

But if the sum of two quantities and the sum of their reciprocals is known, so is their product and hence their difference and finally the quantities themselves. Thus

$$\frac{g_m}{h_m} \quad \text{and} \quad \frac{g_n}{h_n} \qquad\qquad \text{are functions of } u.$$

Now A may be written in the form

$$A = -\frac{h_m h_n}{\Delta \cdot \Delta} = -\frac{1}{\Delta}\left(\frac{1}{g_n/h_m - g_n/h_n}\right),$$

which shows that Δ is itself a function of u, and hence from the original forms of A, B, C, it follows that

$$(16) \qquad h_m h_n, \; g_m g_n, \; g_m h_n, \; g_n h_m \qquad \text{are functions of } u \text{ or } g.$$

It may be noted that in the derivation of these conditions the assumption has tacitly been made that none of the four quantities g_m, g_n, h_m, h_n, which occur in denominators, vanish. In case g_m or g_n should vanish there would be no need of a transformation to equipotentials. In case h_m or h_n should vanish a similar discussion of A, B, C shows that the conditions (16) are none the less satisfied. It may further be noted that of the four conditions only three, in particular the last three, are independent. Furthermore, the function g may be replaced by V or F which are functions of g: the more general function g has been introduced to cover the cases in which some function of F or V might be simpler to treat than F or V. The results may be stated as

THEOREM 1. *The conditions that the extremals of the problem*

$$(1) \qquad \int F(x, y)\, ds, \qquad ds = \sqrt{dx^2 + dy^2}, \text{ a minimum}$$

may be found by direct quadrature by transformation to the equipotentials are that the three quantities

$$(16') \qquad g_m g_n, \qquad g_m h_n, \qquad g_n h_m, \qquad m \text{ and } n \text{ given by (12)},$$

where g is any convenient function of F, be functions of F.

3. **Further discussion of the conditions.** The expression of the fact that the functional determinants of each of the three functions (16′) and g vanish is

$$\begin{vmatrix} g_{mn} g_n + g_m g_{mn} & g_m \\ g_{mn} g_n + g_m g_{nn} & g_n \end{vmatrix} = 0 \qquad \text{or} \qquad g_n^2 g_{mm} = g_m^2 g_{nn},$$

$$\begin{vmatrix} g_{nm} h_n + g_m h_{mn} & g_m \\ g_{mn} h_n + g_m h_{nn} & g_n \end{vmatrix} = 0, \qquad \begin{vmatrix} g_{mn} h_m + g_n h_{mm} & g_m \\ g_{nn} h_m + g_n h_{mn} & g_n \end{vmatrix} = 0,$$

which by the introduction of the auxiliary function $\psi = g_n / g_m$ may be reduced to the three equations

$$(17) \quad g_n^2 g_{mm} - g_m^2 g_{nn} = 0, \qquad \psi h_{mn} - h_{nn} - \psi_m h_n = 0, \qquad \psi h_{mn} - \psi^2 h_{mm} + \psi_n h_m = 0.$$

This leads to a restatement of the conditions in a modified form in the

THEOREM 2. *The conditions that the extremals of* (1) *may be found by direct quadrature by transformation to the equipotentials is that g or F or V shall satisfy the partial differential equation*

$$(17a) \qquad \frac{g_{mm}}{g_m^2} = \frac{g_{nn}}{g_n^2} \qquad \text{or} \qquad \frac{\partial}{\partial m}\left(\frac{1}{g_m}\right) = \frac{\partial}{\partial n}\left(\frac{1}{g_n}\right)$$

and that a function h may be found to satisfy the equations

$$(17b) \qquad \psi h_{mn} - h_{nn} - \psi_m h_n = 0, \qquad \psi h_{mn} - \psi^2 h_{mm} + \psi_n h_m = 0,$$

subject to the restriction that $\Delta = g_m h_n - g_n h_m \neq 0$ and with $\psi = g_n / g_m$.

It may be noted that the function ψ may really be introduced because it was seen that no restriction would result from assuming that neither g_m nor g_n vanish. It is also evident that if g is a solution of (17a), so is any function of g. Furthermore certain solutions of this equation are known, namely,

$$(18) \qquad g = \Phi(am + bn + c) \qquad \text{and} \qquad g = \Phi[(m + a)(n + b)],$$

Φ an arbitrary function, a, b, c arbitrary constants, which correspond to the cases of a force parallel to a line or acting toward a center. As there are two equations (17b), it is not to be expected that they may be satisfied unless the function ψ which occurs in the coefficients is restricted; and it is hardly probable that this restriction is no more than equivalent to (17a). It may turn out, however, that in a given example not even the necessary condition (17a) is satisfied. For instance, in the previous communication in the Annals of Mathematics the potential function was of the form $x + cy^2$, which is immediately seen to fail in satisfying this condition. The transformation to equipotentials would therefore be useless.

To bring in the conditions (17b), the first equation may be differentiated with respect to m and multiplied by ψ and then added to the derivative of the second with respect to n. The derivatives of the third order drop out and so do those of the second order if h_{mm} be substituted from the second of the equations and the result reduced by the relation $\psi\psi_m + \psi_n = 0$ which is an equivalent of (17a). The resulting equation

$$(19) \qquad \psi\psi_{mm} h_n + (2\psi\psi_m^2 - \psi_{nn}) h_m = 0$$

may be added to the two in (17b) and must hold simultaneously with them. If

ψ_{mm} be expressed in terms of g, the result after discarding factors which are not zero is

$$\psi_{mm} \propto g_n g_{m^2n} - g_m g_{mn^2}.$$

That the coefficient of h_m may be obtained by interchanging m and n in this form of ψ_{mm} is obvious from the symmetry of the original conditions (16)*. Hence (19) becomes

$$(19') \qquad (g_n g_{m^2n} - g_m g_{mn^2})(h_n - h_m) = 0.$$

This condition splits into two. Either $\psi_{mm} \propto g_n g_{m^2n} - g_m g_{nn^2} = 0$ or $h_n = h_m$. In the first case ψ may be obtained at once in the form

$$\psi = mf_1(n) + f_2(n)$$

and after substitution in $\psi\psi_m + \psi_n = 0$, which was seen to be the equivalent of (17a), gives the two equations

$$f_1^2 + f_1' = 0, \qquad f_1 f_2 + f_2' = 0.$$

If f_1 and f_2 be determined from these equations the value of ψ is

$$\psi = \frac{m+a}{n+b} = \frac{g_n}{g_m} \qquad \text{or} \qquad g = \Phi[(m+a)(n+b)].$$

Thus in this case there is nothing new: for the value of g is that already determined for central forces. In the second case where $h_n = h_m$, a reference to the original conditions (16) shows that h_n and h_m are each individually functions of g and hence g_m and g_n are functions of g. The functional detrminants in question reduce to the differential equations

$$\frac{g_{mm}}{g_m} = \frac{g_{mn}}{g_n} \qquad \text{and} \qquad \frac{g_{nm}}{g_m} = \frac{g_{nn}}{g_n}$$

or

$$\frac{\partial}{\partial m} \log g_m = \frac{\partial}{\partial m} \log g_n \qquad \text{and} \qquad \frac{\partial}{\partial n} \log g_m = \frac{\partial}{\partial n} \log g_n.$$

From the latter form it follows at once that

$$\psi = \frac{g_n}{g_m} = \text{const.} \qquad \text{or} \qquad g = \Phi(am + bn + c),$$

which again is not new, merely the case of a parallel field.

The results may therefore now be summed up in the general

THEOREM A. *The cases of central and parallel field are the only cases in which a transformation to the equipotentials and to any other system of curves will enable the integration to be carried out immediately owing to the lack of one of the variables in the integrand* Fds.

* The complete work was, however, used as a check on this statement.

It therefore appears that the quest of cases in which the discussion of a string with a free end is other than trivial must be along a less restricted direction. Before attacking this problem, I will make a digression in the following article.

4. On the integration of $p^2t - q^2r = 0$. The integration of equation (17a), although not required in the discussion of the problem here treated, offers some points of interest owing partly to the fact that it belongs to the difficult class of differential equations of the second order and of degree higher than the first, partly to the method employed. In the usual notation the equation is

$$\left(\frac{\partial z}{\partial x}\right)^2 \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial z}{\partial y}\right)^2 \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \psi\psi_x + \psi_y = 0 \quad \text{or} \quad \psi p + q = 0 \quad \text{if} \quad \psi = \frac{\partial z}{\partial y}\Big/\frac{\partial z}{\partial x}.$$

If Charpit's method be applied to $\psi\psi_x + \psi_y = 0$, the resulting system of equations is

$$\frac{dy}{-1} = \frac{dp}{p^2} = \frac{dq}{pq} = \frac{d\psi}{-p\psi - q} = \frac{dx}{-z}.$$

The integration of the first two of this system gives $p^{-1} = y + b$ and hence

$$d\psi = \frac{dx}{y+b} - \frac{\psi dy}{y+b} \qquad \text{or} \qquad dx = yd\psi + \psi dy + bd\psi$$

becomes immediately integrable with the result

$$\psi = \frac{x+a}{y+b} = \frac{\partial z}{\partial y}\Big/\frac{\partial z}{\partial x}, \quad \text{compare } z = \Phi\left[(x+a)(y+b)\right],$$

which is a first complete integral and corresponds precisely with the cases of a central force in the problem above. The first general integral would be obtained by eliminating b from the two equations

$$\psi = \frac{x + \theta(b)}{y+b}, \qquad \frac{1}{\psi} = \theta'(b).$$

On the other hand the integration of the second two equations of the given system gives $bp - aq = 0$ and hence

$$\psi = \frac{b}{a} = \frac{\partial z}{\partial y}\Big/\frac{\partial z}{\partial x}, \quad \text{compare } z = \Phi(ax + by + c),$$

which again is a first complete integral and corresponds to the case of parallel forces. The first general integral here cannot be found by elimination.

The elimination is not readily carried out in case θ is an arbitrary and not an assigned function. On the other hand the first general integral of the equation may be found by interchanging the dependent variable ψ with one of the vari-

ables x, y by means of the well-known formula

$$\left(\frac{\partial\psi}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_\psi\left(\frac{\partial y}{\partial\psi}\right)_x=-1; \qquad \text{whence} \qquad \psi=-\frac{\psi_y}{\psi_x}=\left(\frac{\partial x}{\partial y}\right)_\psi,$$

and

$$x=y\psi+f_1(-\psi)=y\frac{z_y}{z_x}+f_1\left(-\frac{z_y}{z_x}\right)=-y\left(\frac{\partial x}{\partial y}\right)_z+f_1\left(\frac{\partial x}{\partial y}\right)_z.$$

If λ be introduced to represent $(\partial x/\partial y)_z$, the final solution of the equation is found on eliminating λ from the two equations

$$x+\lambda y=f_1(\lambda), \qquad \sqrt{\lambda}y+f_2(z)=\int\frac{f_1'(\lambda)d\lambda}{2\sqrt{\lambda}}.$$

There appears to be no way of obtaining the general solution in a form in which no elimination is necessary.

5. On the reduction of the equation of extremals to the type $y''=0$. If the equation which determines the extremals is to admit an infinitesimal transformation, the set of four equations (8) or (8′) must have a common solution for ξ and η. The method of determining the conditions for a common solution is stated very clearly by TRESSE.* In the first place the given equations should be written in a canonical form by solving for certain of the derivatives of highest order, here the second; then the equations should be differentiated until there are enough equations to solve for all the derivatives of the resulting highest order. In this case one differentiation will suffice to allow a solution for the derivatives of the third order.

$$(20)\quad \begin{aligned}
\eta_{m^3}&=\phi_m\eta_{mm}+\phi_{mm}\eta_m,\\
\eta_{m^2n}&=\phi_m\eta_{mn}+\phi_{mn}\eta_m,\\
\eta_{mn^2}&=-\phi_n\eta_{mn}-\phi_{nn}\eta_m+\phi_{mn}\eta_n+\phi_{mn}\xi_m+\phi_{mn^2}\eta+\phi_{m^2n}\xi,\\
\eta_{n^3}&=(\phi_n^2-2\phi_{nm})\eta_n+3\phi_{mn}\xi_n+(\phi_n\phi_{nn}-\phi_{n^3})\eta+(\phi_n\phi_{mn}-\phi_{mn^2})\xi,\\
\xi_{m^3}&=3\phi_{mn}\eta_m+(\phi_{m^2}-2\phi_{mm})\xi_m+(\phi_m\phi_{mn}-\phi_{m^2n})\eta+(\phi_m\phi_{mm}-\phi_{m^3})\xi.\\
\xi_{m^2n}&=-\phi_m\xi_{mn}+\phi_{mn}\eta_n-\phi_{mm}\xi_n+\phi_{mn}\xi_m+\phi_{mn^2}\eta+\phi_{m^2n}\xi,\\
\xi_{mn^2}&=\phi_n\xi_{mn}+\phi_{mn}\xi_n,\\
\xi_{n^3}&=\phi_n\xi_{n^2}+\phi_{n^2}\xi_n.
\end{aligned}$$

Next the conditions of integrability

$$\eta_{m^2\cdot n}=\eta_{m^2n\cdot m},\ \eta_{m^2n\cdot n}=\eta_{mn^2\cdot m},\ \cdots,\ \xi_{mn^2\cdot n}=\xi_{n^2\cdot m}$$

must be expressed. In this case there are six such conditions. of which two

* Acta Mathematica, vol. 18 (1894), p. 9.

vanish identically and four are equal in pairs. There remain then

$$(21)\quad \begin{aligned} &A(2\xi_m + \eta_n) - B\eta_m + A_n\eta + A_m\xi = 0, \qquad && A = \phi_m\phi_{mn} - \phi_{m^2n},\\ &B(2\eta_n + \xi_m) - A\xi_n + B_n\eta + B_m\xi = 0, \qquad && B = \phi_n\phi_{mn} - \phi_{mn^2}. \end{aligned}$$

These equations are to be joined with (8′) and (20) and further conditions of integrability are to be sought.

Before taking this matter up, it may be noted that if (21) are satisfied identically, the conditions of integrability are certainly fulfilled and no further work is necessary. This will be the case if $A = B = 0$. A reference to the work of R. LIOUVILLE * will show that these conditions are the same as his $L_1 = L_2 = 0$, which are the necessary and sufficient conditions that the equation (4′) should be reducible by a point transformation to the form $y'' = 0$. The conditions are also identical with TRESSE's $H = 0$ which ensures the same possibility of reduction to $y'' = 0$.† It should be noted that $A = B = 0$ if $\phi_{mn} = 0$, that is, if ϕ or $\log F$ satisfies Laplace's equation. It is therefore possible to state

THEOREM 3. *If* $\log F$ *satisfies Laplace's equation* $\Delta \log F = 0$, *that is, if*

$$(22)\qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log F = \frac{\partial^2}{\partial m\,\partial n}\log F = 0, \qquad F = f_1(m)f_2(n),$$

the extremals may be transformed by a suitable transformation $\bar{x} = X(x, y)$, $\bar{y} = Y(x, y)$ *into the* ∞^2 *straight lines of the plane.*

If $\phi_{mn} \neq 0$ while $A = B = 0$, the equations $A = 0$ and $B = 0$ may be put in the form

$$\frac{\partial}{\partial m}\log\phi_{mn} = \frac{\partial}{\partial m}\phi \qquad \text{and} \qquad \frac{\partial}{\partial n}\log\phi_{mn} = \frac{\partial}{\partial n}\phi.$$

From this it is an immediate consequence that

$$(23)\qquad \phi_{mn} = \kappa e^{\phi}, \qquad \frac{\partial^2}{\partial m\,\partial n}\log F^2 = \kappa F^2 \qquad (\kappa \text{ a constant}),$$

and the integral of this equation may be written in the form ‡

$$(24)\qquad F = \gamma\frac{\sqrt{f_1'(m)f_2'(n)}}{f_1(m) + f_2(n)} \qquad \text{where} \qquad \gamma^2\kappa = 2.$$

This result cannot be regarded as a general case under which (22) may fall, although the differential equation (23) from which it is derived may be regarded as the general case of which LAPLACE's is a special case obtained by setting $\kappa = 0$. Both results may be included, however, in the general

* An immediate consequence of the work on pp. 218–219 of the first memoir cited.

† *Jablonowski Preisschrift*, l. c., p. 56.

‡ JORDAN, *Cours d'analyse*, vol. 3, p. 360.

THEOREM B. *The necessary and sufficient conditions that the extremals of* $\int F ds$ *be defined by a differential equation reducible by a transformation* $\bar{x} = X(x, y)$, $\bar{y} = X(x, y)$ *to the form* $y'' = 0$ *are that*

$$\frac{\partial^2}{\partial m\, \partial n} \log F^2 = \kappa F^2, \tag{23}$$

where κ *is any constant (including* $\kappa = 0$*); or that*

$$(24') \quad F = \gamma \frac{\sqrt{f_1'(m) f_2'(n)}}{f_1(m) + f_2(n)}, \qquad \kappa \neq 0 \qquad \text{or} \qquad F = f_1(m) f_2(n), \qquad \kappa = 0,$$

where f_1 *and* f_2 *are arbitrary functions of* m *and* n *respectively.*

6. The case of Laplace's equation. If $\phi_{mn} = 0$, the solution for ϕ may be written in the form of the sum of two functions $g(m)$, $h(n)$

$$\phi = g(m) + h(n) + a + b, \tag{25}$$

where the constants have been added for convenience inasmuch as ϕ enters in the equation (4′) of the extremals only through its derivatives. The set of equations (8′) which determine ξ and η then becomes

$$\begin{gathered} \eta_{mm} = g'(m)\eta_m, \qquad \xi_{nn} = h'(n)\xi_n, \\ \eta_{nn} - 2\xi_{mn} + h'\eta_n - 2g'\xi_n + h''\eta = 0, \\ \xi_{mm} - 2\eta_{mn} + g'\xi_m - 2h'\eta_m + g''\xi = 0. \end{gathered} \tag{26}$$

The first two of these equations may be integrated completely in the form

$$\eta = N\int e^g dm + \bar{N}, \qquad \xi = M\int e^h dn + \bar{M}, \tag{27}$$

where M and $\bar{M}$ are functions of m only and N and $\bar{N}$ functions of n only. If these values be substituted in the third equation of the set, the result may be arranged in the following manner:

$$e^{-h}(N'' + h'N' + h''N)\int e^g dm + e^{-h}(\bar{N}'' + h'\bar{N}' + h''\bar{N}) = 2(M' + g'M), \tag{28}$$

with a similar equation from the fourth of the set.

As (28) is an identity for all values of m and n and as $\int e^g dm$ and e^{-h} may without loss of generality be considered as not identically constant, it follows that the three equations

$$\begin{gathered} M' + g'M = 0, \\ N'' + h'N' + h''N = 0, \\ \bar{N}'' + h'\bar{N}' + h''\bar{N} = 0, \end{gathered} \tag{29}$$

must each be true, with three similar equations from the fourth equation of (26).

The first equation with its analogue yield the solutions

$$(30) \qquad M = ce^{-g}, \qquad N = ke^{-f}$$

for M and N. To determine $\overline{M}$ and $\overline{N}$ there remain the equations

$$(29') \qquad \overline{M}'' + g'\overline{M}' + g''\overline{M} = 0, \qquad \overline{N}'' + h'\overline{N}' + h''\overline{N} = 0.$$

If it be noted that e^{-g} and e^{-f} respectively are particular solutions of these equations, the general solutions may be found as

$$(30') \qquad \overline{M} = e^{-g}\left(c_1 \int e^g\, dm + c_2\right), \qquad \overline{N} = e^{-h}\left(k_1 \int e^h\, dn + k_2\right).$$

On substituting these values of M, $\overline{M}$, N, $\overline{N}$ the expressions for ξ and η are

$$(31) \qquad \begin{aligned} \xi e^g &= c\int e^h\, dn + c_1 \int e^g\, dm + c_2, \\ \eta e^h &= k\int e^g\, dm + k_1 \int e^h\, dn + k_2. \end{aligned}$$

Thus in case $\phi_{mn} = 0$, the exact form of the coefficients ξ, η of Uf may be found. The group is clearly one of eight parameters. The path curves are given by the differential equations

$$\frac{dm}{\xi} = \frac{dn}{\eta} = \frac{e^g\, dm}{c\int e^h\, dn + c_1\int e^g\, dm + c_2} = \frac{e^h\, dn}{k\int e^g\, dm + k_1\int e^h\, dn + k_2},$$

which may be written in the form

$$(32) \qquad \frac{k}{2} G^2 + k_1 GH + k_2 G = \frac{c}{2} H^2 + c_1 GH + c_2 H + \text{const.},$$

where the notation $G = \int e^g\, dm$, $H = \int e^h\, dn$ has been introduced. Moreover in this case the integral of (4′) may be found directly. For

$$\frac{n''}{n'} + h'n' - g' = 0 \qquad \text{or} \qquad d\log n' + h'dn - g'dm = 0,$$

$$\log n' + h - g = C, \qquad n' = Ce^{g-h},$$

and

$$e^h\, dn = Ce^g\, dm \qquad \text{or} \qquad H = CG + K,$$

where C and K are constants. The value of F^2 is $F^2 = Ae^g e^h$.

Now that the computations have been completed it will be advantageous to introduce the new functions $\bar{g} = e^g$, $\bar{h} = e^h$ in terms of which to state the results in

THEOREM 4. *The necessary and sufficient conditions that the differential equation* (4′) *of the extremals be an exact differential equation is that the function* F *be defined by* $F^2 = A\bar{g}\bar{h}$, *in which case the solution of the equation is*

$$(33) \qquad C_1 H + C_2 G = K, \qquad G = \int \bar{g}\, dn, \qquad H = \int \bar{h}\, dn,$$

and the infinitesimal transformations which the equation admits are

$$(31') \qquad \xi\bar{g} = cH + c_1 G + c_2, \qquad \eta\bar{h} = kG + k_1 H + k_2.$$

To obtain real solutions ad libitum the conjugate harmonic functions R and S may be introduced so that

$$\bar{g} = R + Si, \qquad \bar{h} = R - Si, \qquad F^2 = A(R^2 + S^2).$$

Formulated as a theorem, the result may be stated in

THEOREM 5. *If* $F^2 = A(R^2 + S^2)$, *where* R *and* S *are conjugate harmonic functions, the equation* (4) *of the extremals is integrable as*

$$(33') \qquad C_1 \int (R dx - S dy) + C_2 \int (R dy + S dx) = K.^*$$

7. Concerning other cases. The next case to consider would be that of equation (23) in which $\kappa \neq 0$. Here it appears impossible to carry through the determination of the infinitesimal transformations and of the solution of the differential equation of the extremals in terms of the functions f_1 and f_2 which enter in F. Both LIE and R. LIOUVILLE have shown that in case any differential equation is reducible to the form $y'' = 0$, the solution of the given equation may be made to depend upon the solution of a linear differential equation of the third order.† The specialization which their results undergo when applied to the particular equation (4′) does not appear to be sufficient to render the equation of the third order integrable in terms of quadratures. For further details concerning this case reference is therefore made to the original sources.

Thus far the discussion has been wholly on the assumption that the given equation (4′) admits a group of eight parameters. To derive the conditions, which will be in the shape of partial differential equations, that ϕ must satisfy in order that (4′) shall admit a group of one or two or three parameters is an exceedingly tedious task at computation, whether pursued by the differentiation of (21) and the comparison with (8′) to obtain further conditions of integrability or attacked by the more systematic methods of differential invariants as indicated by TRESSE or R. LIOUVILLE.‡ Moreover, the partial differential equa-

* That this formula satisfies (3) or (4) may be seen at once on eliminating C_1, C_2, K by differentiation are on substituting $R_x = S_y$, $R_y = -S_x$, the conditions for conjugate harmonic functions.

† LIE, in a memoir quoted as Norw. Arch., 8 (1883), p. 372, in the Mathematical Encylopedia and as Archives Norwégiennes, 8 (1883), by TRESSE. I have not had access to this reference. LIOUVILLE, in the first memoir cited. LIOUVILLE's reduction is made under certain restrictive hypotheses, see p 240, which are ample, however, to cover the case of equation (4′).

‡ TRESSE in both memoirs, especially the latter, where the restriction of the third degree is not imposed. See references on p. 426, viz. Acta Mathematica, p. 76, and *Jablonowski Preisschrift*, pp. 60–84. LIOUVILLE, in both memoirs, especially the second. LIOUVILLE does not determine the special conditions relative to groups of two or three parameters as distinct from groups of one parameter, but this classification can be imposed if desired.

tions which would be obtained would be of so high an order and degree as to be, in all probability, of no very great value or interest when obtained. It appears therefore that although TRESSE has obtained in an elegant manner and stated in concise form the necessary and sufficient conditions that a given equation of the second order shall admit a group of one or two or three parameters, yet for practical purposes there is no method of telling whether an assigned equation does or does not admit a group.

In simple cases it is frequently possible to set up and actually solve the set of simultaneous partial differential equations for ξ and η which arise from the application of LIE's condition (7) to a given equation; and in such cases as I have tried it appears that this method of actual solution, which will give the values of ξ and η if there are any, is no longer, if as long, as the methods of differential invariants. For instance, there are five equations,

$$
\begin{aligned}
&y'' = 6y^2 + x,\\
&y'' = 2y^3 + xy + \alpha,\\
(34)\qquad &y'' = \frac{y'^2}{y} + e^x(1 - y^2),\\
&y' = \frac{y'^2}{y} + e^x(\alpha y^2 + 1) - e^{2x}y^3,\\
&y'' = \frac{y'^2}{y} + e^x(\alpha y^2 + \beta) + e^{2x}\left(\frac{1}{y} - y^3\right),
\end{aligned}
$$

of particular interest owing to the fact shown by PAINLEVÉ, that they are the only equations of the second order $y'' = R(y', y, x)$, where R is rational in y', algebraic in y, and analytic in x, which have fixed critical points and which define functions that are uniform over the entire complex plane and are *new* transcendents. *

The substitution of $\omega = 6y^2 + x$ in (7) gives the four equations

$$
\xi_y = 0, \qquad \eta_{yy} - 2\xi_{xy} = 0,
$$
$$
2\eta_{xy} - \xi_{xx} - (18y^2 + 3x)\xi_y = 0,
$$
$$
\eta_{xx} + (\eta_y - 2\xi_x)(6y^2 + x) - \xi - 12\eta y = 0.
$$

The first two may be solved completely for ξ and η to give the results

$$
\xi = f(x)y + g(x), \qquad \eta = f'(x)y^2 + h(x)y + k(x).
$$

The substitution of these values in the third equation yields

$$
4f''(x)y + 2h'(x) - f''(x)y - g''(x) - (18y^2 + 3x)f(x) = 0
$$

* PAINLEVÉ, *Sur les equations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme*, Acta Mathematica, vol. 25 (1902), pp. 1–85. See § 11, p. 13.

or

$$f(x) = 0, \qquad 2h'(x) - g''(x) = 0.$$

The additional equations obtainable from the fourth equation are

$$h(x) + 2g'(x) = 0, \quad k''(x) - g(x) + x[h(x) - 2g'(x)] = 0, \quad h''(x) - 12k(x) = 0,$$

and a combination of these with the foregoing shows that $f = g = h = k = 0$. Hence $\xi = 0$ and $\eta = 0$; and the first equation of (34) has no group. The procedure for the second equation of (34) is quite similar.

The result of substitution from PAINLEVÉ's third equation in (7) is

$$\xi_{yy} + \frac{1}{y}\xi_y = 0, \qquad \xi_y = \frac{f(x)}{y}, \qquad \xi = f(x)\log y + g(x),$$

$$\eta_{yy} - \frac{2f'(x)}{y} - \left(\frac{\eta}{y}\right)_y = 0, \qquad \eta_y - 2f'(x)\log y - \frac{\eta}{y} = h(x),$$

$$\eta = yf'(x)(\log y)^2 + yh(x)\log y + yk(x),$$

from setting the coefficients of y'^3 and y'^2 equal to zero; and from the rest

$$-3\xi_y e^x(1 - y^2) + 2\eta_{xy} - \xi_{xx} - 2\frac{\eta_x}{y} = 0,$$

$$(\eta_y - 2\xi_x)e^x(1 - y^2) + \eta_{xx} - \xi e^x(1 - y^2) + 2\eta y e^x = 0.$$

The substitution of the values of ξ and η in these equations and the identification of the coefficient of each power of y to zero shows that here again $\xi = \eta = 0$. And the two remaining cases are similar; in fact the forms of ξ and η as determined from the coefficients of y'^3 and y'^2 are the same. Hence

THEOREM C. *Painlevé's five equations admit no groups.*

This result, which was merely derived incidentally to showing how the direct determination of the problem without the use of invariants is frequently possible, is interesting, though certainly not unexpected. It may be noted, however, that this absence of a group is not a characteristic of equations which define new uniform transcendents in PAINLEVÉ's sense. Such equations may admit relatively simple groups.* Moreover, the groups which he employs are algebraic in y, and it is conceivable that even though his canonical equations admitted no such group, they might still admit some group transcendental in y. Theorem C states that the five types of the second order do not.

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* PAINLEVÉ, loc. cit., § 55, pp. 80–81.